Math 010 Exam 1
Spring 2026

For full credit: Please show work using techniques from this course and use correct mathematical notation.

1. (7 pts) Determine conditions (if any) that a, b, and c must satisfy for the linear system to be consistent.

$$x_1 + 2x_2 - x_3 = a$$
$$2x_1 + 3x_2 + x_3 = b$$
$$3x_1 + 5x_2 = c$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 2 & 3 & 1 & b \\ 3 & 5 & 0 & c \end{array}\right]$$

$R_2 \Rightarrow R_2 - 2R_1$

$$\begin{array}{cccc} 2 & 3 & 1 & b \\ -2 & -4 & 2 & -2a \\ \hline 0 & -1 & 3 & b-2a \end{array}$$

$R_3 \Rightarrow R_3 - 3R_1$

$$\begin{array}{cccc} 3 & 5 & 0 & c \\ -3 & -6 & 3 & -3a \\ \hline 0 & -1 & 3 & c-3a \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & a \\ 0 & -1 & 3 & b-2a \\ 0 & -1 & 3 & c-3a \end{array}\right]$$

$R_3 \Rightarrow R_3 - R_2$

$$\begin{array}{cccc} 0 & -1 & 3 & c-3a \\ 0 & 1 & -3 & -b+2a \\ \hline 0 & 0 & 0 & -a-b+c \end{array}$$

The system is consistent if $\boxed{a+b=c.}$

2. (6 pts) The augmented matrix for a linear system of equations has been reduced to reduced row echelon form. Express the solution set as a linear combination of column vectors that contain only numerical entries.

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 2 & 0 & 5 \\ 0 & 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{array}\right]$$

R_1: $x_1 = -3x_2 - 2x_4 + 5$

R_2: $x_3 = x_4 + 3$

R_3: $x_5 = -2$

Let $s = x_2$, $t = x_4$

$$\vec{x} = \begin{bmatrix} -3s-2t+5 \\ s \\ t+3 \\ t \\ -2 \end{bmatrix} = \boxed{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 5 \\ 0 \\ 3 \\ 0 \\ -2 \end{bmatrix}}$$

3. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 4 & 1 \\ -2 & 3 \end{bmatrix}$. Compute the indicated expression or say why the operation is not defined.

a. (3 pts) AB

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 2+2 & -1+6 & 0-4 \\ 6+0 & -3+0 & 0+0 \\ -2+4 & 1+12 & 0-8 \end{bmatrix} = \begin{bmatrix} 4 & 5 & -4 \\ 6 & -3 & 0 \\ 2 & 13 & -8 \end{bmatrix}$$

b. (3 pts) $tr(BA)$

$$BA = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 2-3+0 & 4+0+0 \\ 1+9+2 & 2+0-8 \end{bmatrix}$$

$$tr(BA) = \underline{-1-6} = -7$$

OR $tr(BA) = tr(AB) = -7$ from part (a)

c. (2 pt) $CB + A$

Not defined. CB is $(2 \times 2)(2 \times 3)$ and so is 2×3. A is 3×2, so CB and A are not the same size.

4. (4 pts) a. Let $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$. Find A^{-1}.

$\det(A) = 6 - 5 = 1$

$A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

b. (2 pts) Use your answer from part (a) to solve the linear system.

$3x_1 + 5x_2 = 2$
$x_1 + 2x_2 = 1$

$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 - 5 \\ -2 + 3 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

5. (8 pts) Use the inversion algorithm to determine whether A is invertible or singular. You don't need to find the inverse; stop working when you can answer the question with certainty.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 2 & 3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - 2R_1$

$$\begin{array}{cccccc} 2 & 3 & 1 & 0 & 0 & 1 \\ -2 & -4 & 0 & -2 & 0 & 0 \\ \hline 0 & -1 & 1 & -2 & 0 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\begin{array}{cccccc} 0 & -1 & 1 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ \hline 0 & 0 & 4 & -2 & 1 & 1 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 4 & -2 & 1 & 1 \end{array}\right]$$

$R_2 \to 4R_2 - 3R_3$

$$\begin{array}{cccccc} 0 & 4 & 12 & 0 & 4 & 0 \\ 0 & 0 & -12 & 6 & -3 & -3 \\ \hline 0 & 4 & 0 & 6 & 1 & -3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 6 & 1 & -3 \\ 0 & 0 & 4 & -2 & 1 & 1 \end{array}\right]$$

Since each row has a pivot,

A is invertible

6. a. (4 pts) Given $B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 7 \end{bmatrix}$, find an elementary matrix E such that $EB = A$.

R_3 of A is $3R_2 + R_3$ of B, so

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

b. (2 pts) Compute the product EB to verify your answer.

$$EB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 7 \end{bmatrix} = A \checkmark$$

7. (3 pts) Let $M = \begin{bmatrix} a & 3 \\ b+1 & 5 \end{bmatrix}$. Find all values of a and b so that M is a symmetric matrix.

a can be any number.

$b+1 = 3 \Rightarrow b = 2$

8. Determine whether each statement is true or false and justify your answer (the justification can be one or two sentences or a counterexample, as appropriate).

a. (3 pts) If a linear system has more unknowns than equations, then it must have infinitely many solutions.

False.

Counter example: $\begin{aligned} 2x+3y+5z &= 4 \\ 2x+3y+5z &= 8. \end{aligned}$

b. (3 pts) If A and B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$.

True. An invertible matrix commutes with its inverse.

9. (8 pts) Prove that if A is an invertible $n \times n$ matrix, then the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

A is invertible $\Rightarrow A^{-1}$ exists.

Then $A\vec{x} = \vec{0} \Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{0}$

$\Rightarrow \vec{x} = \vec{0}$.

If $\vec{x}_1$ is any other solution, then

$A\vec{x} - A\vec{x}_1 = \vec{0} - \vec{0}$

$\Rightarrow A(\vec{x} - \vec{x}_1) = \vec{0} \Rightarrow \vec{x} - \vec{x}_1 = \vec{0}$

$\Rightarrow \vec{x} = \vec{x}_1$.

$\vec{x} = \vec{0}$ is the only solution. ✓

10. (8 pts) Prove that if A is an invertible $n \times n$ matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

A invertible $\Rightarrow A^{-1}$ exists.

$$(A^T)(A^{-1})^T = (A^{-1}A)^T$$
$$= I^T$$
$$= I.$$

OR

$$AA^{-1} = I$$
$$(AA^{-1})^T = I^T = I$$
$$(A^{-1})^T A^T = I$$

Since multiplication of A^T by $(A^{-1})^T$ results in the identity matrix, $(A^T)^{-1}$ exists and $(A^T)^{-1} = (A^{-1})^T$. ✓